Triangle Qubit Channel

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Given a triangle on a unit circle, we proved that an unital qubit channel can be constructed by cosine correlation functions. Furthermore, we proved that Bell's inequality is non-violation for the antipodal point of the triangle qubit channel. Finally, we discussed a triangle qubit channel formed by three cosine wave functions with relatively prime frequencies.

I. INTRODUCTION

Quantum channels can transmit quantum states and classical information. In particular, a qubit channel is a quantum channel with a single qubit. Quantum channels play an important role in quantum computing, quantum communication, and quantum cryptography. In physics, we can implement quantum channels by the transmission of entangled photons through fiber optics or free space.

This paper originated from the author's previous research on Jones polynomials in quantum computing[1] and on Bell's inequality[2] of polynomial matrix[3]. In theorem 1, we constructed a qubit channel based on a triangle on the unit circle and figured out the cosine correlation functions as its elements.

To theorem 2, from K. Martin[4]'s research on the scope of quantum channels and the research of A. Fujiwara and P. Algoet[5] on Fujiwara-Algoet Condition (FAC), we proved the non-violation of Bell's inequality[2] for the antipodal point of an unital qubit channel. We also refered to the research of D. Braun[6] and colleagues on universal features of the quantum channels included the M.-D. Choi's matrix[7] for completely positive and trace-preserving.

II. BACKGROUND

Define $\mathfrak{M}_d(\mathbb{C})$ to be the set of $d \times d$ matrices over the complex field $\mathbb{C}$ and $\mathfrak{D}_d(\mathbb{C})$ to be the set of $d \times d$ density operator matrices that is positive, Hermitian, and trace one. A channel $\phi : \mathfrak{M}_d(\mathbb{C}) \to \mathfrak{M}_d(\mathbb{C})$ is a linear map that is completely positive with trace-preserving. A channel ϕ is also a map $\phi : \mathfrak{D}_d(\mathbb{C}) \to \mathfrak{D}_d(\mathbb{C})$. A unitary channel $\phi_u : \mathfrak{M}_d(\mathbb{C}) \to \mathfrak{M}_d(\mathbb{C})$ is the set $\phi_u(\rho) = U\rho U^\dagger$ where U is a unitary $d \times d$ matrix and the operator $\rho \in \mathfrak{D}_d(\mathbb{C})$ is a density operator matrix.[6]

A qubit channel $\phi : \mathfrak{M}_2(\mathbb{C}) \to \mathfrak{M}_2(\mathbb{C})$ is a two-dimensional channel. We can represent any state $\rho \in \mathfrak{M}_2(\mathbb{C})$ by Pauli matrices as the basis such that $\rho = \frac{1}{2}\sum_{i=0}^{3} r_i\sigma_i$ where $r_i \in \mathbb{R}$ with $r_0 = 1$. The $trace(\rho) = 1$ and $r = (r_1, r_2, r_3)$ is the Bloch vector. A qubit channel ϕ acting on state $\rho \in \mathfrak{M}_2(\mathbb{C})$ is a 4 by 4 real homogeneous matrix T_ϕ. The positivity of Choi's matrix of T_ϕ is equivalent to the complete positivity of the qubit channel ϕ.

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An unital qubit channel is a qubit channel ϕ such that $\phi(I/2) = I/2$, that is the matrix $T_\phi = diag(1, \lambda_1, \lambda_2, \lambda_3)$. An antipodal point or an antipode of a point in a Bloch sphere is the diametrically opposite point. The antipodal map $\phi' : \mathfrak{B}_3(\mathbb{R}) \to \mathfrak{B}_3(\mathbb{R})$ in Bloch sphere is $\phi'(\lambda) = -\lambda$.

III. TRIANGLE QUBIT CHANNEL

Theorem 1. *Given a triangle on the unit circle, there exists an unital qubit channel described by the triple* $(\lambda_1, \lambda_2, \lambda_3) = (cos(2\pi\omega_1)cos(2\pi\omega_3),\ cos(2\pi\omega_2)cos(2\pi\omega_3),\ cos(2\pi\omega_1)cos(2\pi\omega_2))$ *where the* $\omega_1, \omega_2, \omega_3 \in \mathbb{R}$.

Proof. Three vertices on the unit circle of the complex plane make a unique triangle. Given these three distinct vertices as a triple $(exp(i2\pi\theta_1), exp(i2\pi\theta_2), exp(i2\pi\theta_3))$ where $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$, we define the triangle qubit matrix $T(\Theta; K_4)$ with the basis of the Klein four-group K_4 below,

$$T(\Theta; K_4) = \frac{1}{2}\left(\kappa_0 + \kappa_1 exp(i2\pi\theta_1) + \kappa_2 exp(i2\pi\theta_2) + \kappa_3 exp(i2\pi\theta_3)\right) \tag{1}$$

where $\Theta = (\theta_1, \theta_2, \theta_3)$ and $K_4 = (\kappa_0, \kappa_1, \kappa_2, \kappa_3)$. Below is the K_4 representation with four diagonal matrices that their elements are in {-1,0,+1} and κ_0 is the identity,

$$\kappa_0 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \kappa_1 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2}$$

$$\kappa_2 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \kappa_3 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \tag{3}$$

where $det(\kappa_i) = 1$, $trace(\kappa_i) = 0 (i \neq 0)$, and κ_i is diagonal for all $i \in \{0, 1, 2, 3\}$. The absolute square of the triangle qubit $T(\Theta, K_4)$ is

$$\begin{aligned} & T(\Theta, K_4)T(\Theta, K_4)^* = T(\Theta, K_4)T(\Theta, K_4)^\dagger \\ =& \frac{1}{2}(\kappa_0 + \kappa_1 exp(+i2\pi\theta_1) + \kappa_2 exp(+i2\pi\theta_2) + \kappa_3 exp(+i2\pi\theta_3)) \\ *& \frac{1}{2}(\kappa_0 + \kappa_1 exp(-i2\pi\theta_1) + \kappa_2 exp(-i2\pi\theta_2) + \kappa_3 exp(-i2\pi\theta_3)). \end{aligned}$$

Since K_4 is an Abelian group with κ_0 as identity, ${\kappa_i}^2 = I_4$ for all $i \in \{0, 1, 2, 3\}$, and $\kappa_1\kappa_2 = \kappa_3, \kappa_1\kappa_3 = \kappa_2, \kappa_2\kappa_3 = \kappa_1$, the absolute square of $T(\Theta,K_4)$ is equal to

$$\begin{aligned}\kappa_0+&\frac{1}{4}\{\kappa_1(exp(i2\pi\theta_1)+exp(-i2\pi\theta_1))\\&+\kappa_2(exp(i2\pi\theta_2)+exp(-i2\pi\theta_2))\\&+\kappa_3(exp(i2\pi\theta_3)+exp(-i2\pi\theta_3))\\&+\kappa_1(exp(i2\pi(\theta_2-\theta_3))+exp(-i2\pi(\theta_2-\theta_3)))\\&+\kappa_2(exp(i2\pi(\theta_1-\theta_3))+exp(-i2\pi(\theta_1-\theta_3)))\\&+\kappa_3(exp(i2\pi(\theta_1-\theta_2))+exp(-i2\pi(\theta_1-\theta_2))\}.\end{aligned}$$

Furthermore, let $\hat{\theta}=\frac{\theta_1+\theta_2+\theta_3}{2}$, by Eluer's formula and cosine multiplication formula, the absolute square of $T(\Theta,K_4)$ is equal to

$$\begin{aligned}\kappa_0+&\kappa_1cos(\hat{\theta}-\theta_3)cos(\hat{\theta}-\theta_2)+\kappa_2cos(\hat{\theta}-\theta_3)cos(\hat{\theta}-\theta_1)\\&+\kappa_3cos(\hat{\theta}-\theta_2)cos(\hat{\theta}-\theta_1).\end{aligned}$$

Moreover, let $\omega_1,\omega_2,\omega_3\in\mathbb{R}$ such that $\theta_1=\omega_1+\omega_3,\theta_2=\omega_2+\omega_3,\theta_3=\omega_1+\omega_2$, we obtain $\hat{\theta}=\omega_1+\omega_2+\omega_3,\omega_1=\hat{\theta}-\theta_2,\omega_2=\hat{\theta}-\theta_1$, and $\omega_3=\hat{\theta}-\theta_3$, the absolute square of $U(\Theta,K_4)$ is equal to

$$\begin{aligned}&\kappa_0+\kappa_1cos(2\pi\omega_1)cos(2\pi\omega_3)+\kappa_2cos(2\pi\omega_2)cos(2\pi\omega_3)\\&\quad+\kappa_3cos(2\pi\omega_1)cos(2\pi\omega_2)\end{aligned}$$

Let $(\lambda_1,\lambda_2,\lambda_3)=(cos(2\pi\omega_1)cos(2\pi\omega_3),cos(2\pi\omega_2)cos(2\pi\omega_3),cos(2\pi\omega_1)cos(2\pi\omega_2))$ and $(q_0,q_1,q_2,q_3)=\frac{1}{4}(1+\lambda_1+\lambda_2+\lambda_3,1+\lambda_1-\lambda_2-\lambda_3,1-\lambda_1+\lambda_2-\lambda_3,1-\lambda_1-\lambda_2+\lambda_3)$, since $T(\Theta;K_4)T(\Theta;K_4)^\dagger\geq 0$ which is equivalent to $diag(q_0,q_1,q_2,q_3)\geq 0$ where the $\frac{1}{4}(q_0,q_1,q_2,q_3)$ are eigenvalues of Choi's matrix[7] for density operator $\rho=\frac{1}{2}(I_2+\sum_{i=1}^3\lambda_i\sigma_i)$. In addition, since $|\lambda_i|\leq 1$, by Lemma 5.11[4] and Choi's theorem[7], the diagonal matrix $diag(1,\lambda_1,\lambda_2,\lambda_3)$ is a unital qubit channel. □

Remark 1. *By the definition of* $(\lambda_1,\lambda_2,\lambda_3)$ *above, there is* $\lambda_1\lambda_2\lambda_3=(cos(2\pi\omega_1)cos(2\pi\omega_2)cos(2\pi\omega_3))^2$, *thus* $cos(2\pi\omega_1)cos(2\pi\omega_2)cos(2\pi\omega_3)=\pm\sqrt{\lambda_1\lambda_2\lambda_3}$. *We obtain* $cos(2\pi\omega_1)=\pm\sqrt{\lambda_1\lambda_3/\lambda_2}$, $cos(2\pi\omega_2)=\pm\sqrt{\lambda_2\lambda_3/\lambda_1}$, *and* $cos(2\pi\omega_3)=\pm\sqrt{\lambda_1\lambda_2/\lambda_3}$. *The map from* λ_i *to* ω_i *is not bijective.*

IV. ANTIPODAL POINT AND BELL'S INEQUALITY

Theorem 2. *The antipodal point of a triangle qubit channel on Bloch sphere is non-violation Bell's inequality with cosine correlation functions.*

Proof. Given a qubit channel with a point $(\lambda_1,\lambda_2,\lambda_3)$ in Bloch sphere, its antipodal point is equal to $(\lambda_1',\lambda_2',\lambda_3')=(-\lambda_1,-\lambda_2,-\lambda_3)$ with the density operator $\rho'=\frac{1}{2}(I_2+\sum_{i=1}^3\lambda_i'\sigma_i)$. Choi's matrix of ρ' is

$$C_{\phi'}=\frac{1}{4}\begin{pmatrix}+1-\lambda_3&0&0&-\lambda_1-\lambda_2\\0&+1+\lambda_3&-\lambda_1+\lambda_2&0\\0&-\lambda_1+\lambda_2&+1+\lambda_3&0\\-\lambda_1-\lambda_2&0&0&+1-\lambda_3\end{pmatrix}. \quad (4)$$

Let the eigenvalues of $C_{\phi'}$ be (q_0',q_1',q_2',q_3'), we obtain $q_0'=\frac{1}{4}(1-\lambda_1-\lambda_2-\lambda_3)$, $q_1'=\frac{1}{4}(1-\lambda_1+\lambda_2+\lambda_3)$, $q_2'=\frac{1}{4}(1+\lambda_1-\lambda_2+\lambda_3)$, $q_3'=\frac{1}{4}(1+\lambda_1+\lambda_2-\lambda_3)$. To the corresponding q_i in the proof of Theorem 1, for all $i\in\{0,1,2,3\}$ there are $q_i+q_i'=\frac{1}{2}$ and $0\leq q_i\leq 1$ because $|\lambda_i|\leq 1$ and q_i is an eigenvalue of Choi's matrix of the unital qubit channel. Thus, we have $-\frac{1}{2}\leq q_i'\leq\frac{1}{2}$.

To an unital qubit channel, there are $q_1=1+\lambda_1-\lambda_2-\lambda_3\geq 0$ and $q_2=1-\lambda_1+\lambda_2-\lambda_3\geq 0$. That is, $-(1-\lambda_3)\leq\lambda_1-\lambda_2\leq 1-\lambda_3$ which is $|\lambda_1-\lambda_2|\leq 1-\lambda_3$, one of a FAC inequality[5]. For the antipodal point since $\lambda_i'=-\lambda_i$, we have $|\lambda_1'-\lambda_2'|\leq 1+\lambda_3'$. When.$P_{xz},P_{yz},P_{xy}$ are correlation functions and $\lambda_1'=P_{xz}$, $\lambda_2'=P_{yz}$, $\lambda_3'=P_{xy}$, we obtain Bell's inequality[2]:

$$|P_{xz}-P_{yz}|\leq 1+P_{xy}. \quad (5)$$

Especially, let the cosine correlation functions be $P_{xz}=cos(2\pi\omega_1)cos(2\pi\omega_3)$, $P_{yz}=cos(2\pi\omega_2)cos(2\pi\omega_3)$, and $P_{xy}=cos(2\pi\omega_1)cos(2\pi\omega_3)$, we proved the theorem for the antipodal point. □

V. DISCUSSION

A special case in theorem 1 is $(\omega_1,\omega_2,\omega_3)=(pt,qt,rt)$ where (p,q,r) are relatively prime and $t\in\mathbb{R}$, we obtain three cosine wave functions $cos(2\pi pt)$,$cos(2\pi qt)$, and $cos(2\pi rt)$ with frequencies (p,q,r). The triangle qubit matrix $T(\Theta;K_4)$ becomes

$$T(z,p,q,r;K_4)=\frac{1}{2}(\kappa_0+\kappa_1z^{p+r}+\kappa_2z^{q+r}+\kappa_3z^{p+q}) \quad (6)$$

where $z=exp(i2\pi t)$, which is a polynomial matrix over Klein four-group. In physics, we can apply three cosine waves with zero phases to construct a continuous triangle qubit channel.

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