Triangle Qubit Channel, Fidelity and Teleportation

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Triangle qubit channels are a type of quantum channel used in the context of quantum communication and quantum information processing. In this article, we describe a method for constructing and analyzing triangle qubit channels, and derive explicit formulas for measuring the fidelity of these channels using qubit fidelity and concurrence as functions of Bloch vectors and mixed-state inputs. We also show the link between the fidelity of these channels and the double-slit experiment in physics, and demonstrate how triangle qubit channels can be used in qubit teleportation. Moreover, since the fidelity of parallel qubits is a bivariate surface, we compute its gradient, Hessian matrix, and Gaussian curvature and derive a quasilinear partial differential equation, a Monge-Ampere equation, where fidelity is their solution. In advance, we discuss the use of the Positive Partial Transpose (PPT) criterion and Sylvester's criterion to derive the entanglement expression for the Bell Diagonal State, and use a teleportation gate to calculate the fidelity for single, chain, and loop configurations of teleportation gates. Finally, we describe how to generate a periodic Werner state as a simple entanglement resource by setting a single parameter for triangle qubit channels.

I. INTRODUCTION

A quantum channel is a complete positive and trace-preserving (CPTP) linear map between two density matrices. [3] proved the equivalent of the positivity of the Choi matrix and the complete positivity of a quantum channel. A qubit channel is a quantum channel for qubits. The unital qubit channel is able to represented as a diagonal matrix [9] and [2].

The fidelity of a quantum channel measures the similarity between two quantum states [7]. It is defined as the overlap between the initial and final states of the quantum system, where the final state is the state of the quantum system after it has been transmitted through the channel. The concurrence is a measure of the entanglement of a quantum state. It is a measure of the amount of quantum correlation between two qubits. For qubits, there exist explicit computable fidelity formulas [6]. Fidelity and concurrence are two conjugation elements to measure qubits [16].

Quantum teleportation is a quantum state transmission via classical channels and maximally entangled states [1]. Under a noisy environment, it is hard to generate and transmit maximally entangled states for teleportation. Hence, we need to make non-maximally entangled states with high fidelity [9]. To analyze entanglement of mixed states, we can apply the Positive Partial Transpose (PPT) criterion [14] [5]. A teleportation circuit shows the procedure of teleportation explicitly in the textbook [13]. Bell Diagonal States (BDS) and the Werner States are states for quantum teleportation and communication [8], [4].

In this article, we construct an unital qubit channel with a set of cosine correlation functions. For any mixed state input, we analyze the qubit fidelity and concurrence, which are six-dimensional functions. To parallel Bloch vectors, the fidelity is a two-variable surface. Thus, we can calculate its Gradient, Hessian, and Gaussian curvature. In addition, the fidelity is the solution of the quasilinear partial differential equation or the Monge-Ampere equation [15]. Furthermore, we find that the fidelity of parallel qubits is related to the superposition and interference of wave functions.

To apply the triangle qubit channel on teleportation, we use the unital qubit channel to generate the Bell Diagonal State (BDS), go through the teleportation gate, and get the output. We obtain the fidelity formulas for single teleportation, a teleportation chain, and a teleportation loop.

Finally, use the triangle qubit channels for teleportation, we construct a periodic Werner state and show that its fidelity is close to 1 under the low-frequency cosine function. The fidelity surface shows a flat band-like shape on the time axis. In particular, we observed that the fidelity of the Werner state is equivalent to that of parallel qubits.

This paper is organized as follows. Sec.I is Introduction. Sec.II proved a theorem of Triangle Qubit Channel. Sec.III discussed Fidelity and Concurrence Sec.IV described Quantum Teleportation. Sec.V defined and discussed Periodic Werner States. Sec.VI is Conclusion.

II. TRIANGLE QUBIT CHANNEL

A. Qubit Channel

Define $\mathfrak{M}_d(\mathbb{C})$ to be the set of $d\times d$ matrices over the complex field $\mathbb{C}$, and $\mathfrak{D}_d(\mathbb{C})$ to be the set of $d\times d$ density operator matrices, which are positive, Hermitian, and trace 1. Channel $\phi:\mathfrak{M}_d(\mathbb{C})\to\mathfrak{M}_d(\mathbb{C})$ is a linear map, which is completely positive while preserving trace. Furthermore, Channel ϕ is also a linear map $\phi:\mathfrak{D}_d(\mathbb{C})\to\mathfrak{D}_d(\mathbb{C})$. The unitary channel $\phi_u:\mathfrak{M}_d(\mathbb{C})\to\mathfrak{M}_d(\mathbb{C})$ is the set $\phi_u(\rho)=U\rho U^\dagger$, where U is a unitary $d\times d$ matrix, and the operator $\rho\in\mathfrak{D}_d(\mathbb{C})$ is the density operator matrix [2].

Qubit channel $\phi:\mathfrak{M}_2(\mathbb{C})\to\mathfrak{M}_2(\mathbb{C})$ is a two-dimensional channel. Any state $\rho\in\mathfrak{M}_2(\mathbb{C})$ is represented by the Pauli matrices as the basis such that $\rho=\frac{1}{2}\sum_{i=0}^{3}r_i\sigma_i$ where $r_i\in\mathbb{R}$

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and $r_0 = 1$. The vector $\vec{r} = (r_x, r_y, r_z)$ with $trace(\rho) = 1$ is the Bloch vector. The qubit channel ϕ acting on the state $\rho \in \mathfrak{M}_2(\mathbb{C})$ is a 4 by 4 real matrix T_ϕ. The positivity of the Choi's matrix of T_ϕ is equivalent to the complete positivity of the qubit channel ϕ [3]. The unital qubit channel is a qubit channel ϕ that is the matrix $T_\phi = diag(1, \lambda_1, \lambda_2, \lambda_3)$ such that $\phi(I/2) = I/2$. [10].

B. Triangle Qubit Channel

For any triangle, there exists a unique circumcircle that passes through each of the three vertices of the triangle. So we only need to consider a triangle on the unit circle.

Theorem 1. *Given a triangle on the unit circle, there exists an unital qubit channel described by Bloch vector* $(\lambda_1, \lambda_2, \lambda_3)$ = $(cos(\omega_2)cos(\omega_3)$, $cos(\omega_3)cos(\omega_1)$, $cos(\omega_1)cos(\omega_2))$ *where* $\omega_1, \omega_2, \omega_3 \in \mathbb{R}$.

Proof. Three distinct vertices on the unit circle of the complex plane compose a unique triangle. Given these three vertices as a vector $(e^{-i\theta_1}, e^{-i\theta_2}, e^{-i\theta_3})$ where $\Theta = (\theta_1, \theta_2, \theta_3)$, $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$, define a triangle qubit matrix $T(\Theta; K_4)$

$$T(\Theta; K_4) = \frac{1}{2}\left(\kappa_0 + \kappa_1 e^{i\theta_1} + \kappa_2 e^{i\theta_2} + \kappa_3 e^{i\theta_3}\right), \tag{1}$$

where the Klein four-group $K_4 = (\kappa_0, \kappa_1, \kappa_2, \kappa_3)$ is an Abelian group with κ_0 as identity, $\kappa_i{}^2 = I_4$ for all $i \in \{0, 1, 2, 3\}$, and $\kappa_1\kappa_2 = \kappa_3, \kappa_1\kappa_3 = \kappa_2, \kappa_2\kappa_3 = \kappa_1$. The K_4 representation of diagonal matrices is as follow:

$$\kappa_0 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \kappa_1 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2}$$

$$\kappa_2 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \kappa_3 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \tag{3}$$

where $det(\kappa_i) = 1$, $trace(\kappa_i) = 0$ for $i \neq 0$, and κ_i is diagonal for all $i \in \{0, 1, 2, 3\}$.

The modulus squared of triangle qubit matrix is

$$\begin{aligned} |T(\Theta, K_4)|^2 =& T(\Theta, K_4)^\dagger T(\Theta, K_4) \\ =& \frac{1}{2}(\kappa_0 + \kappa_1 e^{i\theta_1} + \kappa_2 e^{i\theta_2} + \kappa_3 e^{i\theta_3}) \\ &* \frac{1}{2}(\kappa_0 + \kappa_1 e^{-i\theta_1} + \kappa_2 e^{-i\theta_2} + \kappa_3 e^{-i\theta_3}). \end{aligned}$$

Since a Klein four-group is an Abelian group, the modulus squared of $T(\Theta, K_4)$ is equal to

$$\begin{aligned} \kappa_0 + \frac{1}{4}\big\{ & \kappa_1(e^{i\theta_1} + e^{-i\theta_1}) + \kappa_2(e^{i\theta_2} + e^{-i\theta_2}) + \kappa_3(e^{i\theta_3} + e^{-i\theta_3}) \\ & + \kappa_1(e^{i(\theta_2-\theta_3)} + e^{-i(\theta_2-\theta_3)}) + \kappa_2(e^{i(\theta_1-\theta_3)} + e^{-i(\theta_1-\theta_3)}) \\ & + \kappa_3(e^{i(\theta_1-\theta_2)} + e^{-i(\theta_1-\theta_2)})\big\}. \end{aligned}$$

Then, let $\hat{\theta} = \frac{\theta_1+\theta_2+\theta_3}{2}$, and apply Eluer and cosine multiplication formulas, we obtain that the modulus squared of $T(\Theta, K_4)$ is equal to

$$\begin{aligned} \kappa_0 + & \kappa_1 cos(\hat{\theta} - \theta_2)cos(\hat{\theta} - \theta_3) + \kappa_2 cos(\hat{\theta} - \theta_3)cos(\hat{\theta} - \theta_1) \\ & + \kappa_3 cos(\hat{\theta} - \theta_1)cos(\hat{\theta} - \theta_2). \end{aligned}$$

Given $\vec{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ such that $\theta_1 = \omega_2 + \omega_3, \theta_2 = \omega_3 + \omega_1, \theta_3 = \omega_1 + \omega_2$, we obtain $\hat{\theta} = \omega_1 + \omega_2 + \omega_3$, so $\omega_1 = \hat{\theta} - \theta_1, \omega_2 = \hat{\theta} - \theta_2$, and $\omega_3 = \hat{\theta} - \theta_3$, the modulus squared of $U(\Theta, K_4)$ is equal to

$$\kappa_0 + \kappa_1 cos(\omega_2)cos(\omega_3) + \kappa_2 cos(\omega_3)cos(\omega_1) + \kappa_3 cos(\omega_1)cos(\omega_2)$$

Let $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in [-1, +1]^3$ be equal to

$$(cos(\omega_2)cos(\omega_3), cos(\omega_3)cos(\omega_1), cos(\omega_1)cos(\omega_2)) \tag{4}$$

and

$$\begin{aligned} q_0 &= \frac{1}{4}(1 + \lambda_1 + \lambda_2 + \lambda_3), \\ q_1 &= \frac{1}{4}(1 - \lambda_1 + \lambda_2 - \lambda_3), \\ q_2 &= \frac{1}{4}(1 + \lambda_1 - \lambda_2 - \lambda_3), \\ q_3 &= \frac{1}{4}(1 - \lambda_1 - \lambda_2 + \lambda_3), \end{aligned} \tag{5}$$

since $T(\Theta; K_4)^\dagger T(\Theta; K_4) \geq 0$, we have $diag(q_0, q_1, q_2, q_3) \geq 0$. Because (q_0, q_1, q_2, q_3) are eigenvalues of the Choi matrix of the density operator $\rho = \frac{1}{2}(I_2 + \sum_{i=1}^{3} \lambda_i \sigma_i)$ with $|\lambda_i| \leq 1$, by Lemma 5.11 [9] and Choi's theorem [3], the diagonal matrix $diag(1, \lambda_1, \lambda_2, \lambda_3)$ is an unital qubit channel. □

Remark 1. *By previous definition of* $(\lambda_1, \lambda_2, \lambda_3)$, *we have* $\lambda_1\lambda_2\lambda_3 = (cos(\omega_1)cos(\omega_2)cos(\omega_3))^2$, *thus* $cos^2(\omega_1) = \lambda_2\lambda_3/\lambda_1$, $cos^2(\omega_2) = \lambda_1\lambda_3/\lambda_2$, *and* $cos^2(\omega_3) = \lambda_1\lambda_2/\lambda_3$, *the map from* $\vec{\lambda}$ *to* $\vec{\omega}$ *is not bijective. Furthermore,* $0 \leq cos^2(x) \leq 1$ *implies* $\lambda_2\lambda_3 \leq \lambda_1$, $\lambda_3\lambda_1 \leq \lambda_2$, *and* $\lambda_1\lambda_2 \leq \lambda_3$.

III. FIDELITY AND CONCURRENCE

A. Definition of Fidelity and Concurrence

Fidelity measures the similarity between two density operator matrices. Fidelity is defined as [7]

$$F(S_A, S_B) = \left(tr\left(\sqrt{\sqrt{S_A} S_B \sqrt{S_A}}\right)\right)^2. \tag{6}$$

If both S_A *and* S_B *are qubit states, the explicit formula of the fidelity for any mixed states is [7] [6]*

$$F(S_A, S_B) = tr(S_A S_B) + 2\sqrt{det(S_A)det(S_B)}. \tag{7}$$

Let $\tau_1, \tau_2, ...$ *be eigenvalues of the state* $\sqrt{\sqrt{S_A} S_B \sqrt{S_A}}$ *such that* $\tau_1 \geq \tau_2 \geq ...$, *the concurrence of* S_A *and* S_B *is defined as* $C(S_A, S_B) = max(0, \tau_1 - \sum_{k>1} \tau_k)$ *[16].*

To simplify, we express the fidelity and the concurrence squared as $F_1(S_A,S_B) = F(S_A,S_B), F_2(S_A,S_B) = C(S_A,S_B)^2$. Since $F_1(S_A,S_B) + F_2(S_A,S_B) = 2tr(S_A S_B)$ [16], we have

$$F_2(S_A,S_B) = tr(S_A S_B) - 2\sqrt{det(S_A)det(S_B)}. \tag{8}$$

Later, concurrence squared is denoted as concurrence2. Consider the matrix $FC(S_A,S_B) = \begin{pmatrix} F_1(S_A,S_B) & 0 \\ 0 & F_2(S_A,S_B) \end{pmatrix}$, its characteristic equation is

$$z^2 - tr(FC(S_A,S_B))z + det(FC(S_A,S_B)) = 0, \tag{9}$$

where

$$\begin{aligned} tr(FC(S_A,S_B)) &= 2tr(S_A S_B), \\ det(FC(S_A,S_B)) &= (tr(S_A S_B))^2 - 4det(S_A)det(S_B). \end{aligned} \tag{10}$$

$F_{1,2}(S_A,S_B)$ is the maximal/minimal root of the equation respectively.

Remark 2. *Since S_A and S_B are density matrices, their determinants are non-negative real numbers. Thus, the descriminant $\Delta = 16det(S_A)det(S_B) \geq 0$ and two roots are non-negative real numbers. If $\Delta = 0$ then $det(S_A) = 0$ or $det(S_B) = 0$, we obtain $F_1(S_A,S_B) = F_2(S_A,S_B) = tr(S_A S_B)$. The state S_A or S_B is in the pure state if $det(S_A) = 0$ or $det(S_B) = 0$.*

B. Qubit Fidelity and Concurrence2 Formulas

In this subsection, we are going to discuss the explicit formulas for qubit fidelity and concurrence2 of two-qubit states.

Let I_2 be the identity, $\vec{r} = (r_x, r_y, r_z)$, $\vec{s} = (s_x, s_y, s_z)$ be the vector in the Bloch sphere, $r = \|r\|, s = \|s\|$ be norms of the vectors, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ be vector of Pauli matrices, we express qubits S_A and S_B as

$$\begin{aligned} S_A &= \frac{1}{2}(I_2 + \vec{r}\cdot\vec{\sigma}) \qquad & S_B &= \frac{1}{2}(I_2 + \vec{s}\cdot\vec{\sigma}) \\ &= \begin{pmatrix} \frac{1+r_z}{2} & \frac{r_x - ir_y}{2} \\ \frac{r_x+ir_y}{2} & \frac{1-r_z}{2} \end{pmatrix}, & &= \begin{pmatrix} \frac{1+s_z}{2} & \frac{s_x - is_y}{2} \\ \frac{s_x+is_y}{2} & \frac{1-s_z}{2} \end{pmatrix}. \end{aligned} \tag{11}$$

Since $2(det(S_A), det(S_B)) = (1 - tr(S_A^2), 1 - tr(S_B^2))$, we have

$$F_{1,2}(S_A,S_B) = tr(S_A S_B) \pm \sqrt{(1 - tr(S_A^2))(1 - tr(S_B^2))}. \tag{12}$$

In addition, $tr(S_A S_B) = (1 + \vec{r}\cdot\vec{s})/2$, $tr(S_A^2) = (1 + \vec{r}\cdot\vec{r})/2$, and $tr(S_B^2) = (1 + \vec{s}\cdot\vec{s})/2$, we obtain

$$F_{1,2}(S_A,S_B) = \frac{(1 + \vec{r}\cdot\vec{s}) \pm \sqrt{(1-r^2)(1-s^2)}}{2}. \tag{13}$$

By Lagrange's identity $\|\vec{r}\times\vec{s}\|^2 = \|\vec{r}\|^2\|\vec{s}\|^2 - (\vec{r}\cdot\vec{s})^2$, we obtain

$$\begin{aligned} F_1(S_A,S_B) + F_2(S_A,S_B) &= (1 + \vec{r}\cdot\vec{s}), \\ F_1(S_A,S_B) * F_2(S_A,S_B) &= \frac{1}{4}\left(\|\vec{r}+\vec{s}\|^2 - \|\vec{r}\times\vec{s}\|^2\right). \end{aligned} \tag{14}$$

$F_1(S_A,S_B)$ and $F_2(S_A,S_B)$ are the maximal and minimal roots of the equation

$$z^2 - (1 + \vec{r}\cdot\vec{s})z + \frac{1}{4}(\|\vec{r}+\vec{s}\|^2 - \|\vec{r}\times\vec{s}\|^2) = 0. \tag{15}$$

Remark 3. *Since fidelity is not a metric, it is not suitable for some applications. But by fidelity, Bures metric in finite dimension can be defined as Bures angle $D_A(S_A,S_B)$ and Bures distance $D_B(S_A,S_B)$ as follows,*

$$\begin{aligned} D_A(S_A,S_B) &= \arccos\sqrt{F_1(S_A,S_B)}, \\ D_B(S_A,S_B)^2 &= 2(1 - \sqrt{F_1(S_A,S_B)}). \end{aligned} \tag{16}$$

C. Quasilinear Partial Derivative Equations

Generally, let $\vec{x} = (x_1, x_2, ..., x_n)$, $\vec{y} = (y_1, y_2, ..., y_n)$ be n-dimentational vectors, where $x_i, y_i \in [-1, +1]$, n is positive integer, we define the extended fidelity and concurrence2 as

$$F_{1,2}(\vec{x},\vec{y}) = \frac{1}{2}(1 + \vec{x}\cdot\vec{y} \pm \sqrt{(1 - \vec{x}\cdot\vec{x})(1 - \vec{y}\cdot\vec{y})}). \tag{17}$$

Thus we have

$$\begin{aligned} \frac{\partial F_{1,2}(\vec{x},\vec{y})}{\partial x_i} &= \frac{1}{2}(y_i \mp Kx_i), \\ \frac{\partial F_{1,2}(\vec{x},\vec{y})}{\partial y_i} &= \frac{1}{2}(y_i \mp Kx_i)(\frac{-1}{K}), \end{aligned} \tag{18}$$

where $K = \sqrt{\frac{1-y^2}{1-x^2}}$, $x = \|x\|$, $y = \|y\|$, hence, $F_{1,2}(\vec{x},\vec{y})$ satisfied the quasilinear first order PDE,

$$\sqrt{1-x^2}\frac{\partial F_{1,2}(\vec{x},\vec{y})}{\partial x_i} \pm \sqrt{1-y^2}\frac{\partial F_{1,2}(\vec{x},\vec{y})}{\partial y_i} = 0 \tag{19}$$

for all $i \in \{1, ..., n\}$.

The characteristics of the quasilinear Partial Derivative Equation (PDE) are

$$dt = \frac{dx_i}{\sqrt{1-x^2}} = \frac{dy_i}{\pm\sqrt{1-y^2}} = \frac{dF_{1,2}}{0} \tag{20}$$

or

$$\frac{dy_i}{dx_i} = \pm K = \pm\sqrt{\frac{1-y^2}{1-x^2}}. \tag{21}$$

Square and do differential on the first equation of Eq.(20), we have

$$\left(\frac{dx_i}{dt}\right)^2 + \sum_{i=1}^{n} {x_i}^2 = 1, \left(\frac{dy_i}{dt}\right)^2 + \sum_{i=1}^{n} {y_i}^2 = 1 \tag{22}$$

Since $u = dx_1/dt = dx_2/dt ... = dx_n/dt$, $v = dy_1/dt = dy_2/dt ... = dy_n/dt$, we obtain

$$\frac{d^2u}{dt^2} + nu = 0, \frac{d^2v}{dt^2} + nv = 0, (u, v \neq 0) \tag{23}$$

.

They are homogenous Ordinated Derivative Equations (ODEs) or Simple Harmonic Equations. For parametric variable t, the solutions of ODEs are

$$\frac{dx_i}{dt} = u(t) = R_u cos(\omega_u t - \delta_u) \tag{24}$$

with initial condition

$$c_{u,1} = u(0),\ c_{u,2} = \frac{1}{\sqrt{n}} \frac{du}{dt}\bigg|_{t=0} \tag{25}$$

where $\omega_u = \sqrt{n}$ *is frequency,* $R_u = \sqrt{c_{u,1}^2 + c_{u,2}^2}$ *is amplitude,* $tan(\delta_u) = c_{u,2}/c_{u,1}$ *and* δ_u *is phase. The same result is true for v. For fidelity,*

$$\frac{R_v cos(\sqrt{n}t - \delta_v)}{R_u cos(\sqrt{n}t - \delta_u)} = K. \tag{26}$$

Finally, we have

$$\begin{aligned} x_i &= \frac{R_u}{\sqrt{n}} sin(\sqrt{n}t - \delta_u) + c_{x_i}, \\ y_i &= \frac{R_v}{\sqrt{n}} sin(\sqrt{n}t - \delta_v) + c_{y_i}, \end{aligned} \tag{27}$$

where c_{x_i}, c_{y_i} *are constants.*

Remark 4. *For vectors of dimension n, their frequencies* $\omega_x = \omega_y = \sqrt{n}$ *are equal. In particular, n=3 is the dimension of the qubit in the Bloch sphere.*

D. Perpendicular Qubit Fidelity

Two vectors $\vec{r}$ *and* $\vec{s}$ *are perpendicular if and only if* $\vec{r} \cdot \vec{s} = 0$*, then*

$$F_{1,2}(S_A, S_B) = \frac{1}{2}\left(1 \pm \sqrt{(1-r^2)(1-s^2)}\right). \tag{28}$$

Two vectors $\vec{r}$ *and* $\vec{s}$ *being perpendicular means that they are completely non-correlated.*

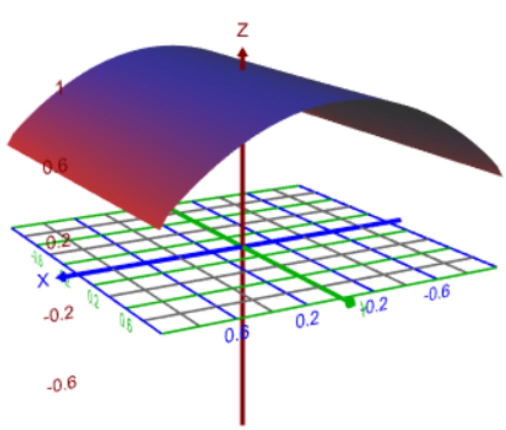


FIG. 1: In the figure, the axis $X = r$, the axis $Y = s$, and the axis $Z = F_1$. Fidelity Surface of Perpendicular Qubit.

E. Parallel Qubit Fidelity

1. Fidelity Surface

Two vectors $\vec{r}, \vec{s} \in [-1,1]^3$ *are parallel if and only if there exists a scalar* $\lambda \in [-1,1]$ *such that* $\vec{s} = \lambda \vec{r}$*, thus*

$$F_{1,2}(S_A, S_B) = \frac{1 + \lambda r^2 \pm \sqrt{\Delta}}{2} \tag{29}$$

where $\Delta = (1-r^2)(1-\lambda^2 r^2))$ *is the discriminant.*
$|\lambda| \le 1$ *and* $r \le 1$ *imply that* $\Delta \ge 0$ *and* $F_{1,2}$ *are real numbers. If* $\Delta = 0$*, then* $r^2 = 1$ *or* $(\lambda r)^2 = 1$*, so* $F_1 = F_2 = (1+\lambda r^2)/2$*. Especially, for* $\lambda = \{-1, 0, +1\}$

$$F_{1,2}(S_A, S_B) = \begin{cases} (1, r^2) & \lambda = +1 \\ \frac{1}{2}(1 \pm \sqrt{1-r^2}) & \lambda = 0 \\ (1-r^2, 0) & \lambda = -1 \end{cases}$$

Let $u = \|r\| = cos(\gamma), v = \lambda\|r\| = cos(\eta)$*, since* $0 \le \|r\| \le 1$*,* $-1 \le \lambda \le 1$*, in the range of* $0 \le \gamma \le \pi/2$*,* $0 \le \eta \le \pi$*,* (u,v) *to* (γ, η) *is a bijective map. Because*

$$cos(arccos(u) \mp arccos(v)) = uv \pm \sqrt{(1-u^2)(1-v^2)} \tag{30}$$

thus,

$$F_{1,2}(\gamma, \eta) = cos^2(\frac{\gamma \mp \eta}{2}) \tag{31}$$

The Bures angle and distance of parallel qubits are

$$\begin{aligned} D_A(S_A, S_B) &= \frac{\gamma - \eta}{2}, \\ D_B^2(S_A, S_B) &= 4sin^2\left(\frac{\gamma - \eta}{4}\right), \end{aligned} \tag{32}$$

where $-\pi/2 \le (\gamma - \eta)/2 \le \pi/4$.

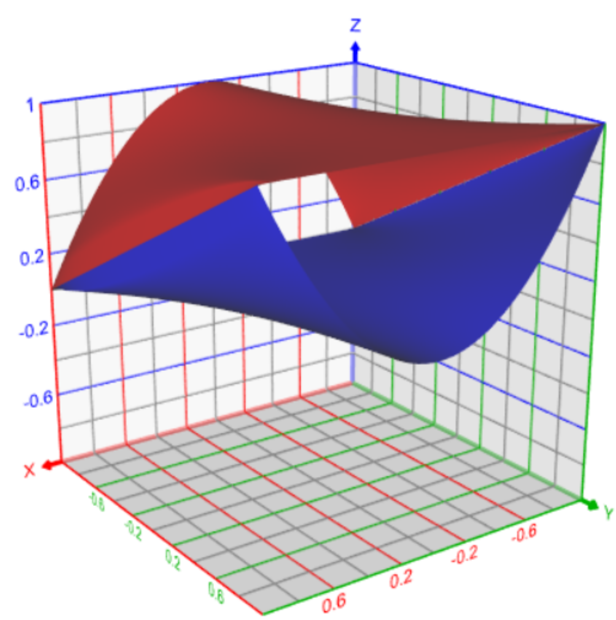


FIG. 2: In the figure, axis $X = r, Y = \lambda, Z = F_{1,2}$. For the parallel vectors $\vec{s} = \lambda\vec{r}$, the fidelity surface (red) is concave, and the *concurrence*^{2} surface (blue) is convex. Both surfaces are smoothly connected.

2. Periodic Parallel Qubit Fidelity

Given $\vec{s} = \lambda\vec{r}$ such that $\lambda = cos^2(kx+\omega t)$ where k is the wavenumber, x is the position, ω is the frequency, and t is the time, the fidelity and concurrence^2 of periodic paralle qubit are represented as

$$F_{1,2}(S_A,S_B) = \frac{1}{2}(1+cos^2(kx+\omega t)r^2 \pm \sqrt{(1-r^2)(1-cos^4(kx+\omega t)r^2)}) \tag{33}$$

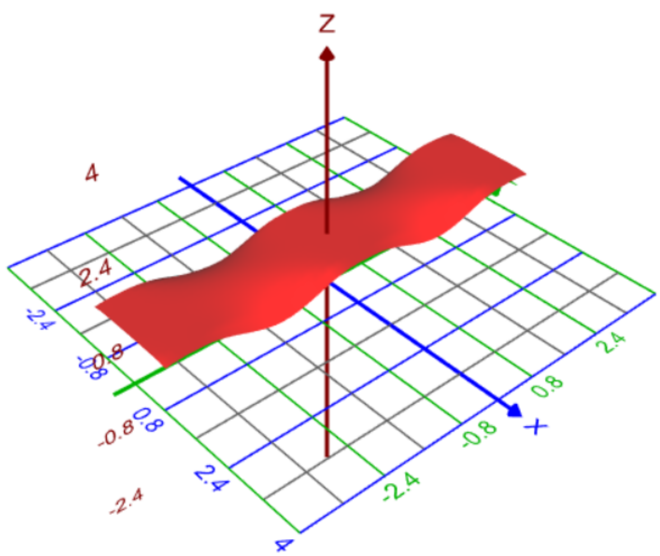


FIG. 3: In the figure, the X axis denotes r, the Y axis denotes t, and the Z axis denotes F_1. Set the frequency $\omega = 2\pi f$, $f = 1/7$, position $x = 0$, the fidelity surface of the periodic parallel qubit looks like a strip of fried bacon floating below the fidelity plane Z=1 a little bit.

3. Hessian Matrix and Monge-Ampere Equation

$F_{1,2}(r,\lambda)$ are graphs of differentiable functions with two variables $r \in [0,1]$ and $\lambda \in [-1,+1]$. The gradients of $F_{1,2}(r,\lambda)$ are vectors of first-order partial derivatives,

$$\nabla F_1(r,\lambda) = \begin{pmatrix} \frac{\partial F_1(r,\lambda)}{\partial r} \\ \frac{\partial F_1(r,\lambda)}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} \frac{-r}{k}(1-k\lambda)^2 \\ \frac{r^2}{2}(1-k\lambda) \end{pmatrix}, \tag{34}$$

where $k = \sqrt{\frac{1-r^2}{1-(\lambda r)^2}}$. Notice that k is not a polyminoal. $F_1(r,\lambda)$ is the solution of quasi-linear partial differential equation,

$$(kr)\frac{\partial F_1(r,\lambda)}{\partial r} + (1-k\lambda)\frac{\partial F_1(r,\lambda)}{\partial \lambda} = 0 \tag{35}$$

Hence, we have the Monge-Ampere equation [15],

$$det(D^2(F(r,\lambda)) = x_1x_2 - (x_1+x_2)\frac{\partial^2 F_1(r,\lambda)}{\partial r \partial \lambda}, \tag{36}$$

where $(x_1,x_2) = (\frac{c_1}{1-b}, \frac{c_2}{a})$ such that $(a,b) = (rk, \lambda k)$,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -\frac{\partial a}{\partial r} & \frac{\partial b}{\partial r} \\ -\frac{\partial a}{\partial \lambda} & \frac{\partial b}{\partial \lambda} \end{pmatrix} \begin{pmatrix} \frac{\partial F_1(r,\lambda)}{\partial r} \\ \frac{\partial F_1(r,\lambda)}{\partial \lambda} \end{pmatrix} \tag{37}$$

and

$$\frac{\partial^2 F_1(r,\lambda)}{\partial r \partial \lambda} = r\left(1 - \lambda k\left(1+\frac{1}{2}\left(\frac{1}{1-\lambda^2 r^2} - \frac{1}{1-r^2}\right)\right)\right). \tag{38}$$

$F_1(r,\lambda)$ is the solution of the Monge-Ampere Equation Eq.(36).

4. Special Points

Points of a surface with zero gradient are called critical points. Eigenvalues and eigenvectors of the Hessian matrix of a surface are their principal curvatures and principal directions of the curvatures, respectively. Gaussian curvature is the product of two principal curvatures. Mean curvature is the average of two principal curvatures.
Let gradient $\nabla F_1(r,\lambda) = (0,0)$, we obtain two critical sets $r = 0$ and $\lambda = 1$. Two Hessian matrices on the critical sets are

$$D^2(F_1(0,\lambda)) = \begin{pmatrix} \frac{-(1-\lambda)^2}{2} & 0 \\ 0 & 0 \end{pmatrix}$$
$$D^2(F_1(r,1)) = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{r^2}{2} \end{pmatrix} \tag{39}$$

Since one eigenvalue of $D^2(F(r,\lambda))$ on the critical sets is zero, others are negative, their $det(D^2(F(r,\lambda))) = 0$. They are degenerate critical points of $F_1(r,\lambda)$. Thus, at the critical point, since $|\nabla F_1(r,\lambda)| = 0$, the Gaussian curvature is equal to the determinant of Hessian matrix and is zero. The mean curvatures is non-positive $-(\frac{1-\lambda}{2})^2$ and $-(\frac{r}{2})^2$. All critical points of fidelity surface are degenerate meaning that the fidelity surface does not belong to the classification of Morse function. [12] [11]

F. Fidelity and Superposition

Consider two wave functions $|\psi_1\rangle$, $|\psi_2\rangle$, they have the same amplitude and global phase difference in Hopf coordinates: $|\psi_1\rangle = cos(\gamma)|0\rangle + e^{i\phi}sin(\gamma)|1\rangle$, $|\psi_2\rangle = cos(\eta)|0\rangle + e^{i\phi}sin(\eta)|1\rangle$, the averages of superposition sum and difference are

$$\frac{(|\psi_1\rangle \pm |\psi_2\rangle)}{2} = \frac{1}{2}((cos(\gamma) \pm cos(\eta))|0\rangle + e^{i\phi}(sin(\gamma) \pm sin(\eta))|1\rangle). \tag{40}$$

The squared amplitudes of the averages of superposition sum and difference are

$$\left|\frac{|\psi_1\rangle \pm |\psi_2\rangle}{2}\right|^2 = cos^2(\frac{\gamma \mp \eta}{2}). \tag{41}$$

Hence, two-qubit fidelity and concurrence^2 $F_{1,2}(\gamma,\eta) = \left|\frac{|\psi_1\rangle \mp |\psi_2\rangle}{2}\right|^2$*, which show the effects of quantum superposition and interference.*
Given $\gamma = cos^{-1}(r)$ *and* $\eta = cos^{-1}(\lambda r)$*, a new expression of the fidelity and concurrence squared is*

$$F_{1,2}(r,\lambda) = cos^2\left(\frac{cos^{-1}(r) \mp cos^{-1}(\lambda r)}{2}\right) \tag{42}$$

In the case of Periodic Werner State with $\lambda = cos^2(2\pi f t)$*,*

$$F_{1,2}(r,t) = cos^2\left(\frac{cos^{-1}(r) \mp cos^{-1}(cos^2(2\pi f t) r)}{2}\right) \tag{43}$$

where f *is the frequency and* t *is the time.*

G. Fidelity and Double-slit Experiment

In the double-slit experiment, the intensity of the wave interference through double slits is $I = I_0 cos^2\left(\frac{\phi}{2}\right)$*, where* ϕ *is the phase shift. It is similar to the fidelity formula Eq.(41) where the meaning of the phase difference is the same. Setting* r *be constant and* $\lambda = r cos^2(\omega)$*, we can generate the interference fringes like that in the double-slit experiment.*

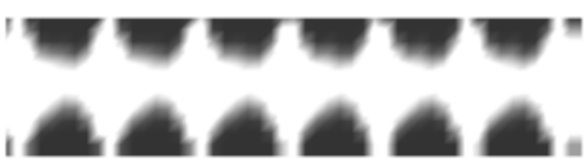


FIG. 4: Fidelity of Periodic Werner State Surface Topview f=5/2.

For the fidelity of the Periodic Werner States, the $\gamma - \eta$ *is the phase difference, the fidelity is same as the intensity when we assign initial intensity* I_0 *to be zero. When we set the* γ *to be a constant, we generate the interference fringes in computer rendering from top-z viewing.*

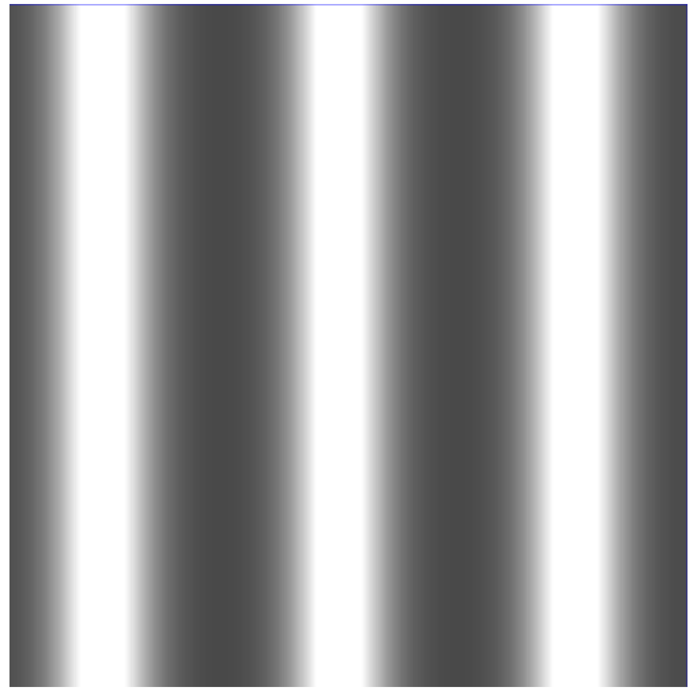
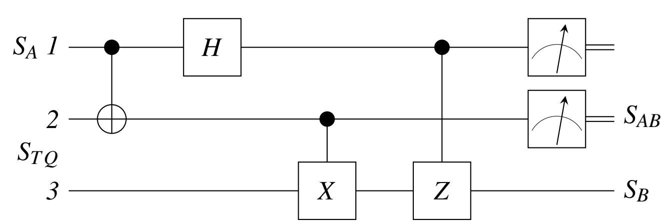


FIG. 5: Fidelity of Periodic Werner State Interference.

IV. QUANTUM TELEPORTATION

By using the shared entangled state S_{TQ} *along with the local operations and classical communication (LOCC), quantum teleportation transfers the input state* S_A *from the sender Alice to the receiver Bob, and obtains the output state* S_B*. For quantum teleportation applications, we use a triangle qubit channel to generate Bell Diagonal State (BDS) as the entangled state* S_{TQ} *shared by Alice and Bob. The diagram below shows the quantum circuit of quantum teleportation [13]. Note that the measurement has moved to the final stage.*



A. Teleportation States

In the diagram, the density matrix S_A *is Alice's input qubit*

$$S_A = \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - i r_y \\ r_x + i r_y & 1 - r_z \end{pmatrix}, \tag{44}$$

where (r_x, r_y, r_z) *is a point in Bloch sphere.* S_B *is Bob's output state;* S_{TQ} *is the entangled state pair shared by Alice and Bob; and* S_{AB} *is the output state of the entire teleportation. The other gates and measurements are processing units.*

B. Generate Bell Diagonal States

We observe that the eigenvalues of an unital qubit channel are non-negative, and their sum is 1, so they can be used as the probability distribution of the Bell basis, forming a Bell Diagonal State (BDS). The permutation of the probability distribution is the S_4 symmetric group with 4!=24 elements.
Bell basis is a collection of Bell states. According to the notation in the textbook [13], we can express the Bell basis as a matrix of $|\beta_{jk}\rangle$,

$$\begin{pmatrix}|\beta_{00}\rangle & |\beta_{01}\rangle \\ |\beta_{10}\rangle & |\beta_{11}\rangle\end{pmatrix} = \begin{pmatrix}\frac{|00\rangle+|11\rangle}{\sqrt{2}} & \frac{|01\rangle+|10\rangle}{\sqrt{2}} \\ \frac{|00\rangle-|11\rangle}{\sqrt{2}} & \frac{|01\rangle-|10\rangle}{\sqrt{2}}\end{pmatrix}, \tag{45}$$

where $|\beta_{jk}\rangle = \frac{1}{\sqrt{2}}(|0,j> + (-1)^j|1,k\oplus 1>)$, $j,k \in \{0,1\}$, and $\oplus$ is the xor. A Bell Diagonal State (BDS) is a two-qubit state that is diagonal in the Bell basis. It is represented as

$$S_{BD} = \sum_{j,k=0}^{1} p_{jk}\,|\beta_{jk}\rangle\langle\beta_{jk}| \tag{46}$$

where $\{p_{jk}\}$ is a probablity distribution with $0 \le p_{jk} \le 1$ and $\sum_{j,k=0}^{1} p_{jk} = 1$. Given an unital qubit channel $diag(1,\lambda_1,\lambda_2,\lambda_3)$ Eq.(5), we have

$$\begin{pmatrix}q_{00}\\q_{01}\\q_{10}\\q_{11}\end{pmatrix} = \frac{1}{4}\begin{pmatrix}+1&+1&+1&+1\\+1&-1&+1&-1\\+1&+1&-1&-1\\+1&-1&-1&+1\end{pmatrix}\begin{pmatrix}1\\\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix},$$

where $\sum_{j,k=0}^{1} q_{jk} = 1$ and $0 \le q_{jk} \le 1$ for all $j,k \in \{0,1\}$. Since

$$|\beta_{00}\rangle\langle\beta_{00}| = \frac{1}{2}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}, |\beta_{01}\rangle\langle\beta_{01}| = \frac{1}{2}\begin{pmatrix}0&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&0\end{pmatrix},$$

$$|\beta_{10}\rangle\langle\beta_{10}| = \frac{1}{2}\begin{pmatrix}1&0&0&\bar{1}\\0&0&0&0\\0&0&0&0\\\bar{1}&0&0&1\end{pmatrix}, |\beta_{11}\rangle\langle\beta_{11}| = \frac{1}{2}\begin{pmatrix}0&0&0&0\\0&1&\bar{1}&0\\0&\bar{1}&1&0\\0&0&0&0\end{pmatrix}, \tag{47}$$

where $\bar{1} = -1$, we obtain the Bell diagnoal state constructed by the triangle qubit channel $S_{TQ} = \sum_{0\le j,k\le 1} q_{jk}\,|\beta_{jk}\rangle\langle\beta_{jk}|$,

$$S_{TQ} = \frac{1}{4}\begin{pmatrix}1+\lambda_1&0&0&\lambda_2+\lambda_3\\0&1-\lambda_1&\lambda_2-\lambda_3&0\\0&\lambda_2-\lambda_3&1-\lambda_1&0\\\lambda_2+\lambda_3&0&0&1+\lambda_1.\end{pmatrix} \tag{48}$$

S_{TQ} can also be expressed as $\frac{1}{4}(I_2\otimes I_2 + \lambda_2(\sigma_1\otimes\sigma_1) - \lambda_3(\sigma_2\otimes\sigma_2) + \lambda_1(\sigma_3\otimes\sigma_3))$.

C. Entanglement of Bell Diagnoal States

According to the Positive Partial Transpose (PPT) criterion [14] [5], for the positive density matrix S_{TQ} in Eq.(48), if and only if exists a negative eigenvalue of the partial transpose matrix $S_{TQ}^{T_B}$, then the two qubits in the state S_{TQ} is entangled.

$$S_{TQ}^{T_B} = \frac{1}{4}\begin{pmatrix}1+\lambda_1&0&0&\lambda_2-\lambda_3\\0&1-\lambda_1&\lambda_2+\lambda_3&0\\0&\lambda_2+\lambda_3&1-\lambda_1&0\\\lambda_2-\lambda_3&0&0&1+\lambda_1\end{pmatrix} \tag{49}$$

The eigenvalues of $S_{TQ}^{T_B}$ are $\frac{1}{4}(1-\lambda_1-\lambda_2-\lambda_3), \frac{1}{4}(1+\lambda_1-\lambda_2+\lambda_3), \frac{1}{4}(1-\lambda_1+\lambda_2+\lambda_3), \frac{1}{4}(1+\lambda_1+\lambda_2-\lambda_3)$. In contrast, the eigenvalues of S_{TQ} are $\frac{1}{4}(1+\lambda_1+\lambda_2+\lambda_3), \frac{1}{4}(1-\lambda_1+\lambda_2-\lambda_3), \frac{1}{4}(1+\lambda_1-\lambda_2-\lambda_3), \frac{1}{4}(1-\lambda_1-\lambda_2+\lambda_3)$. To the state of unital qubit channel S_{TQ} , all the eigenvalues are non-negative, thus, if one of an eigenvalue of $S_{TQ}^{T_B}$ is negative, the state S_{TQ} is entangled.
By Sylvester's criterion, a Hermitian maxtrix such as $S_{TQ}^{T_B}$ is positive-definite if and only if the determinants of the matrices (the upper left i-by-i corner of $4S_{TQ}^{T_B}$, i=1,2,3,4) are positive,

$$det(1+\lambda_1) = 1+\lambda_1 \ge 0, \tag{50}$$

$$det\begin{pmatrix}1+\lambda_1&0\\0&1-\lambda_1\end{pmatrix} = 1-\lambda_1^2 \ge 0 \tag{51}$$

$$\begin{aligned}&det\begin{pmatrix}1+\lambda_1&0&0\\0&1-\lambda_1&\lambda_2+\lambda_3\\0&\lambda_2+\lambda_3&1-\lambda_1\end{pmatrix}\\&=(1+\lambda_1)((1-\lambda_1)^2-(\lambda_2+\lambda_3)^2)\end{aligned} \tag{52}$$

$$\begin{aligned}&det\begin{pmatrix}1+\lambda_1&0&0&\lambda_2-\lambda_3\\0&1-\lambda_1&\lambda_2+\lambda_3&0\\0&\lambda_2+\lambda_3&1-\lambda_1&0\\\lambda_2-\lambda_3&0&0&1+\lambda_1\end{pmatrix}\\&=((1-\lambda_1)^2-(\lambda_2+\lambda_3)^2)((1+\lambda_1)^2-(\lambda_2-\lambda_3)^2)\end{aligned} \tag{53}$$

By Eq.(50) and Eq.(51), the determinants are always non-negative. In Eq.(52), if $|1-\lambda_1| < |\lambda_2+\lambda_3|$ then $S_{TQ}^{T_B}$ is negative thus S_{TQ} is entangled. Otherwise, in Eq.(52), if $|1+\lambda_1| < |\lambda_2-\lambda_3|$ then $S_{TQ}^{T_B}$ is negative S_{TQ} is entangled.

Remark 5. *Formula $|1\mp\lambda_1| < |\lambda_2\pm\lambda_3|$ violates Bell's inequality for $\lambda_1,\lambda_2,\lambda_3$. For S_{TQ}, applying the Positive Partial Transpose (PPT) and Sylvester's criteria, we can derive the Bell's inequality from Bell diagonal states generated by the unital qubit channel. To spacial λ_i relation,*

Remark 6. *Let $\lambda_1 = \lambda_2 = \lambda_3 = \lambda > 0$ or $-\lambda_1 = -\lambda_2 = \lambda_3 = \lambda > 0$, there is one eigenvalue $(1-3\lambda)/4$ and three eigenvalues $(1+\lambda)/4$. Negative eigenvalue means $(1-3\lambda)/4 < 0$. Hence, if $\lambda > 1/3$ then the state S_{TQ} is entangled.*

Remark 7. *By the identity we found below*

$$\begin{aligned}&((1+\lambda_1+\lambda_2+\lambda_3)*(1-\lambda_1+\lambda_2-\lambda_3)*\\&(1+\lambda_1-\lambda_2-\lambda_3)*(1-\lambda_1-\lambda_2+\lambda_3))\\&-((1-\lambda_1-\lambda_2-\lambda_3)*(1+\lambda_1-\lambda_2+\lambda_3)*\\&(1-\lambda_1+\lambda_2+\lambda_3)*(1+\lambda_1+\lambda_2-\lambda_3))\\&=16\lambda_1\lambda_2\lambda_3,\end{aligned} \tag{54}$$

we obtain

$$det(S_{TQ}) - det(S_{TQ}^{T_B}) = \frac{\lambda_1\lambda_2\lambda_3}{16}. \tag{55}$$

Hence, if $det(S_{TQ}^{T_B}) < 0$ *then* $\lambda_1\lambda_2\lambda_3 > 0$.

D. Quantum Teleportation Gate

The teleportation circuit is represented as a unitary gate $U_{TEL} = U_Z^{1,3}U_X^{2,3}U_H^1U_{CNOT}^{1,2}$ *in the diagram, where* $U_Z^{1,3}$ *is the controlled-Z gate wired qubits 1 and 3,* $U_X^{2,3}$ *is the controlled-X gate wired qubits 2 and 3,* U_H^1 *is the Hadamard gate in qubit 1, and* $U_{CNOT}^{1,2}$ *is the CNOT gate wired qubit 1 and 2. The final teleportation matrix is*

$$U_{TEL} = \frac{1}{\sqrt{2}}\begin{pmatrix} +1 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 & 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & +1 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{56}$$

Since $U_{TEL}U_{TEL}^\dagger = U_{TEL}^\dagger U_{TEL} = I$ *and* $det(U_{TEL}) = 1$, U_{TEL} *is a quantum unitary gate. Its eigenvalues are* $(-1,-1,+1,+1,-i,+i,\frac{1+i}{\sqrt{2}},\frac{1-i}{\sqrt{2}})$ *on the unit circle.*

E. Output States

The entire output state is $S_{AB} = U_{TEL}(S_A \otimes S_{TQ})U_{TEL}{}^\dagger$. *Bob's output state is* $S_B = Tr_A(S_{AB})/Tr(S_{AB})$. *After measuring on Alice side, where* $Tr_A(S_{AB})$ *is the partial trace, so*

$$S_B = \frac{1}{2}\begin{pmatrix} 1+\lambda_1 r_z & \lambda_2 r_x - i\lambda_3 r_y \\ \lambda_2 r_x + i\lambda_3 r_y & 1-\lambda_1 r_z \end{pmatrix} \tag{57}$$

The map between S_A *and* S_B *on Bloch spheres is* $(r_x,r_y,r_z) \to (\lambda_2 r_x, \lambda_3 r_y, \lambda_1 r_z)$.

F. Teleportation Fidelity of Bell Diagonal States

The teleportation fidelity and concurrence[2] *of Bell Diagonal States is*

$$F_{1,2}^{TQ}(S_A,S_B) = \frac{1}{2}(1+\lambda_2 r_x^2 + \lambda_3 r_y^2 + \lambda_1 r_z^2 \pm \sqrt{(1-(r_x^2+r_y^2+r_z^2))(1-({\lambda_2}^2 r_x^2 + {\lambda_3}^2 r_y^2 + {\lambda_1}^2 r_z^2))}). \tag{58}$$

If S_A *is in a pure state we have* $r_x^2+r_y^2+r_z^2 = 1$, *otherwise* S_A *is in a mixed state such that* $r_x^2+r_y^2+r_z^2 < 1$. *Using the entangled Bell Diagonal State (BDS) generated by the unital qubit channel, and taking any mixed qubit state as input, we get the teleportation fidelity,*

$$F_{1,2}^{TQ}(\vec{r},\vec{s}) = \frac{1}{2}\left((1+\vec{r}\cdot\vec{s}) \pm \sqrt{(1-r^2)(1-s^2)}\right), \tag{59}$$

where $\vec{\lambda} = (\lambda_1,\lambda_2,\lambda_3)$, $\vec{r} = (r_x,r_y,r_z)$, $P_{(123)}(\vec{\lambda}) = (\lambda_2,\lambda_3,\lambda_1)$ *and* $\vec{s} = P_{(123)}(\vec{\lambda}) \odot \vec{r}$. *The symbol* $\odot$ *represents the Hardmart product of two vectors. The vector* $P_{(123)}(\vec{\lambda})$ *is the cyclic permutation (123) of the vector* $\vec{\lambda}$.

G. Teleportation Chain

We can connect teleporation units as a chain [8]. The n-th state is

$$S_{B_n} = \frac{1}{2}\begin{pmatrix} 1+\lambda_1^n r_z & \lambda_2^n r_x - i\lambda_3^n r_y \\ \lambda_2^n r_x + i\lambda_3^n r_y & 1-\lambda_1^n r_z \end{pmatrix} \tag{60}$$

and the n-th teleportation fidelity is

$$F_{1,2}^{TQ_n} = \frac{1}{2}(1+\vec{r}\cdot\vec{s_n} \pm \sqrt{(1-r^2)(1-\|\vec{s_n}\|^2)}) \tag{61}$$

where $\vec{s_n} = P(\vec{\lambda})^{\odot n} \odot \vec{r}$, $\odot$ *is the Hardmard product of vectors,* $P = (123)$ *is the permutation of vector* $\vec{\lambda}$, *and the n times product* $P(\vec{\lambda})^{\odot n} = P(\vec{\lambda}) \odot P(\vec{\lambda}) \odot \ldots$

H. Teleportation Loop

For the teleportation chain, if $n \to \infty$, *it is a teleportation loop. By squeeze theorem, because* $-|\lambda_i|^n \le \lambda^n \le |\lambda_i|^n$, $0 \le |\lambda_i| < 1$ *for all n, and* $\lim_{n\to\infty}(-|\lambda_i|^n) = \lim_{n\to\infty}(|\lambda_i|^n) = 0$, *thus* $\lim_{n\to\infty}\lambda_i{}^n = 0$ *where* $i \in \{1,2,3\}$.

1. *If* $0 \le |\lambda_i| < 1$ *where* $i \in \{1,2,3\}$, *we get*

$$F_{1,2}^{TQ_\infty} = \lim_{n\to\infty} F_{1,2}^{TQ_n} = \frac{1}{2}\left(1 \pm \sqrt{1-r^2}\right) \tag{62}$$

2. *If one of* $\lambda_i = +1$, $i \in \{1,2,3\}$, *because* $\sum_{i=1}^3 \lambda_i{}^2 \le 1$, *other* $\lambda_j = 0$ *for* $i \ne j$. *Assuming* $i = 1$, *we have*

$$F_{1,2}^{TQ_\infty} = \frac{1}{2}\left(1+r_z^2 \pm \sqrt{(1-r^2)(1-r_z^2)}\right). \tag{63}$$

The output S_{B_∞} *is*

$$S_{B_\infty} = \begin{cases} \frac{1}{2}\begin{pmatrix} 1+r_z & 0 \\ 0 & 1-r_z \end{pmatrix} & \vec{\lambda} = (1,0,0) \\ \frac{1}{2}\begin{pmatrix} 1 & r_x \\ r_x & 1 \end{pmatrix} & \vec{\lambda} = (0,1,0) \\ \frac{1}{2}\begin{pmatrix} 1 & -ir_y \\ ir_y & 1 \end{pmatrix} & \vec{\lambda} = (0,0,1) \end{cases}$$

3. *If one of* $\lambda_i = -1$, $i \in \{1,2,3\}$, *since* $\sum_{i=1}^{3} {\lambda_i}^2 \le 1$, *other* $\lambda_j = 0$ *for* $i \ne j$. *The limit* F_{TQ_∞} *does not exist. The teleportation loop is in the oscillate state and may act as an oscillator.*

   Assuming $i = 1$, *we obtain two sub-sequences that have limits,*

$$F_{1,2}^{TQ_n} = \begin{cases} \frac{1}{2}\left(1 + r_z^2 \pm \sqrt{(1-r^2)(1-r_z^2)}\right) & n \text{ is even} \\ \frac{1}{2}\left(1 - r_z^2 \pm \sqrt{(1-r^2)(1-r_z^2)}\right) & n \text{ is odd} \end{cases}$$

   The output S_{B_∞} *is*

$$S_{B_\infty} = \begin{cases} \frac{1}{2}\begin{pmatrix} 1-r_z & 0 \\ 0 & 1+r_z \end{pmatrix} & \vec{\lambda} = (-1,0,0) \\ \frac{1}{2}\begin{pmatrix} 1 & -r_x \\ -r_x & 1 \end{pmatrix} & \vec{\lambda} = (0,-1,0) \\ \frac{1}{2}\begin{pmatrix} 1 & +ir_y \\ -ir_y & 1 \end{pmatrix} & \vec{\lambda} = (0,0,-1) \end{cases}$$

V. PERIODIC WERNER STATES

A. Werner States

The two-qubit Werner state is defined as a linear combination of a singlet Bell state such as $|\psi\rangle\langle\psi|$ *and the maximally mixed state* I_4,

$$W(\lambda) = \lambda\,|\psi\rangle\langle\psi| + \frac{1-\lambda}{4} I_4 \tag{64}$$

where λ *is a real parameter such that* $0 \le \lambda \le 1$. *Given* $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, *we have* $q_{00} = \frac{1+3\lambda}{4}$, $q_{01} = q_{10} = q_{11} = \frac{1}{4}(1-\lambda)$, *and*

$$W_{TQ}(\lambda) = \sum_{j,k=0}^{1} q_{jk}\,|\beta_{jk}\rangle\langle\beta_{jk}| = \lambda\,|\beta_{00}\rangle\langle\beta_{00}| + \frac{1-\lambda}{4} I_4$$
$$= \frac{1}{4}\begin{pmatrix} 1+\lambda & 0 & 0 & 2\lambda \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 2\lambda & 0 & 0 & 1+\lambda. \end{pmatrix} \tag{65}$$

.
Applying the Werner state W_{TQ} *to the teleportation, we obtain the output state,*

$$S_B = \frac{1}{2}\begin{pmatrix} 1+\lambda r_z & \lambda(r_x - ir_y) \\ \lambda(r_x + ir_y) & 1-\lambda_1 r_z \end{pmatrix} \tag{66}$$

.
The teleportatin fidelity of Werner states is equal to the fidelity of two parallel qubits in Eq.(29). It is

$$F_{W_{TQ}}(r,\lambda) = \frac{1}{2}\left(1 + \lambda r^2 + \sqrt{\Delta}\right). \tag{67}$$

where the discriminant is $\Delta = (1-r^2)(1-\lambda^2 r^2)$.

B. Periodic Werner States

The Werner state of triangle qubit channel is a special parameterization such that $\lambda = \lambda_1 = \lambda_2 = \lambda_3 = cos^2(\omega)$, *thus*

$$W_{TQ}(\omega) = cos^2(\omega)\,|\beta_{00}\rangle\langle\beta_{00}| + \left(\frac{sin(\omega)}{2}\right)^2 I_4. \tag{68}$$

Its teleportation fidelity is

$$F_{W_{TQ}}(r,\omega) = \frac{1}{2}\left(1 + cos^2(\omega) r^2 + \sqrt{(1-r^2)(1-cos^4(\omega) r^2)}\right) \tag{69}$$

where $r \in [0,+1], \omega \in (-\infty,+\infty)$. *For the variable* ω, *it is a periodic function with period* π.

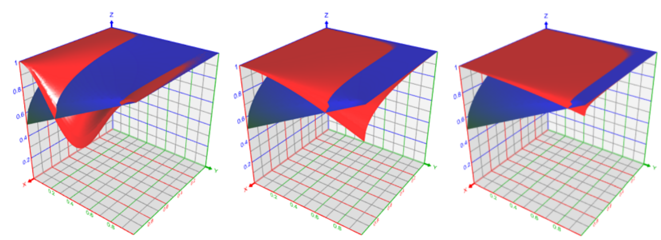


FIG. 6: Blue represents the fidelity of the regular Werner state; red represents the fidelity of the periodic Werner state. The fidelity of Werner states lies in $(x,y,z) = (r,\omega,F(r,\omega)) \in [0,+1] \times (-\infty,+\infty) \times [0,+1]$. Given $\omega = 2\pi f$, figure (a) is f=1/2, (b) is f=1/8, and (c) f=1/16. The lower frequency f shows a flatter fidelity close to the plane (z=1).

Remark 8. $W_{TQ}(\omega)$ *is entangled if* $cos^2(\omega) > 1/3$ *[5]. It is equivalent to* $|cos(\omega)| > 1/\sqrt{3}$. *Assuming* $cos(\omega_0) = 1/\sqrt{3} = 1/tan(\frac{\pi}{3})$ *or* $\omega_0 \approx 54.7356103°$, *we can divide* $\omega \in [0,2\pi)$ *into four intervals:* $[0,\omega_0)$, $[\omega_0, \pi-\omega_0)$, $[\pi-\omega_0, \pi+\omega_0)$, $[\pi+\omega_0, 2\pi)$, *where the* $W_{TQ}(\omega)$ *is in the periodic states of entanglement, separation, entanglement, and separation.*
Let $\omega = 2\pi f * t$, *where* f *is the frequency,* t *is the time, then* $W_{TQ}(\omega)$ *is in the states of periodic entanglement and separation with the time* t.

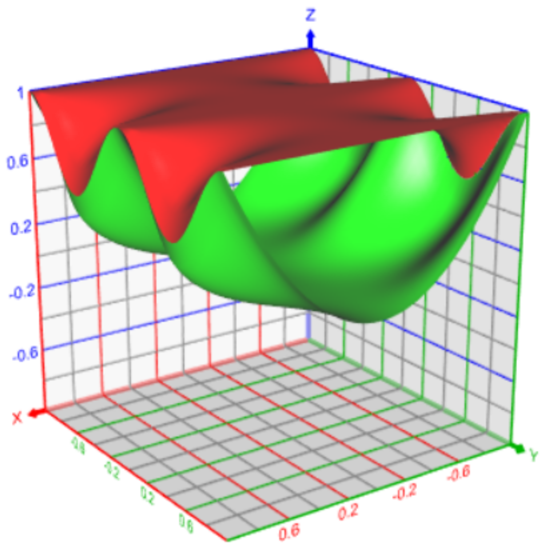


FIG. 7: Blue and red represent the fidelity and concurrence2 of the periodic Werner states, respectively. The domain lies in $(X,Y,Z) = (r,\omega,F(r,\omega)) \in [-1,+1] \times (0,1) \times [0,1]$. Given $\omega = 2\pi f$, the frequency f=1/2.

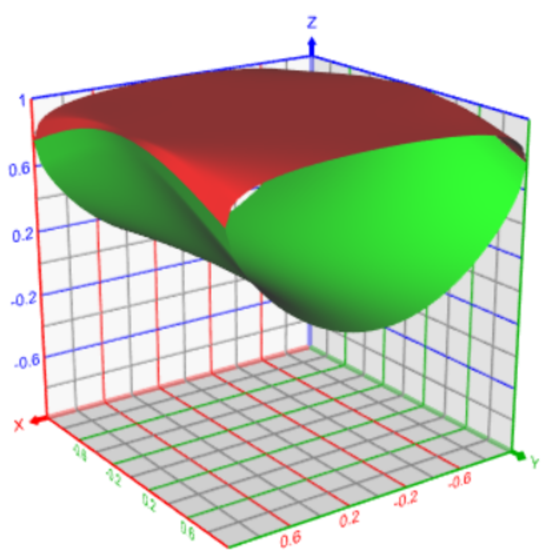


FIG. 8: Blue and red represent the fidelity and concurrence2 of the periodic Werner states, respectively. The domain lies in $(X,Y,Z) = (r,\omega,F(r,\omega)) \in [-1,+1] \times (0,1) \times [0,1]$. Given $\omega = 2\pi f$, the frequency f=1/8.

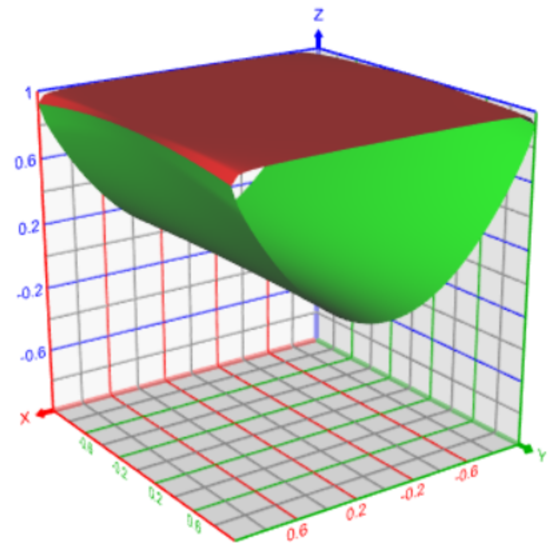


FIG. 9: Blue and red represent the fidelity and concurrence2 of the periodic Werner states, respectively. The domain lies in $(X,Y,Z) = (r,\omega,F(r,\omega)) \in [-1,+1] \times (0,1) \times [0,1]$. Given $\omega = 2\pi f$, the frequency f=1/16. The lower frequency f shows a flatter fidelity close to the plane (z=1).

Remark 9. *The figure below is a top-view of the surface of the periodic Werner state viewed from the positive z-axis. Yellow, blue, green, and red contours use frequency f=1/2, 1/4, 1/8, and 1/16 to enclose the fidelity area. The lower frequency f covers a larger area. This means lower frequencies get higher fidelity.*

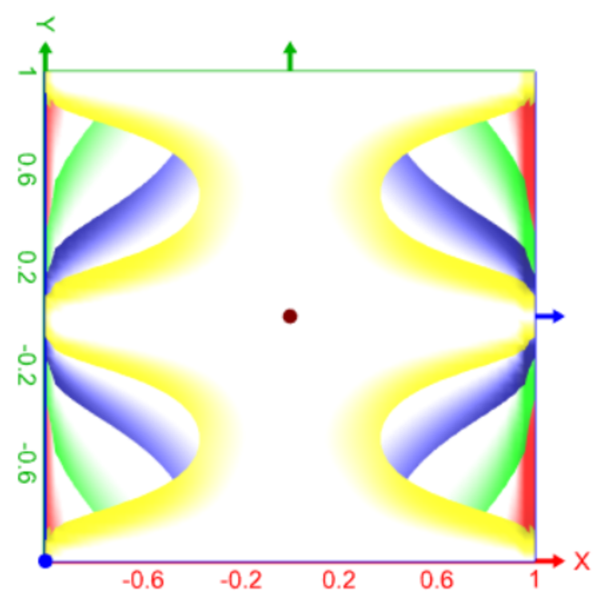


FIG. 10: The figure is the top-view of the surface of periodic Werner states from the positive z-axis where $X = r$ and $Y = \lambda$. The contour yellow, blue, green, and red enclose the fidelity areas with frequencies f=1/2, 1/4, 1/8, and 1/16.

VI. CONCLUSION

In this paper, we developed and studied triangle qubit channels. Besides, we studied qubit fidelity and concurrence squared, and explored their explicit formulas in Bloch space. In addition, we analyzed the geometry of fidelity and concurrence, which are expressed as the roots of a quadratic equation. By creating the Bell diagonal state, we can apply the unital qubit channel to the qubit teleportation. In particular, we developed periodic Werner states constructed from triangle qubit channels.

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